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A classification of the minimal ring extensions of certain commutative rings

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Abstract

All rings considered are commutative with identity and all ring extensions are unital. Let R be a ring with total quotient ring T . The integral minimal ring extensions of R are catalogued via generator-and-relations. If T is von Neumann regular and no maximal ideal of R is a minimal prime ideal of R , the minimal ring extensions of R are classified, up to R -algebra isomorphism, as the minimal overrings (within T) of R and, for maximal ideals M of R , the idealizations $R(+)R/M$ and the direct products $R \times R/M$. If T is von Neumann regular, the minimal ring extensions of R in which R is integrally closed are characterized as certain overrings, up to R -algebra isomorphism, in terms of Kaplansky transforms and divided prime ideals, generalizing work of Ayache on integrally closed domains; no restriction on T is needed if R is quasilocal. One application generalizes a recently announced result of Picavet and Picavet-L'Hermitte on the minimal overrings of a local Noetherian ring. Examples are given to indicate sharpness of the results.

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1. Introduction

All rings and algebras considered below are commutative with identity; and all ring/algebra homomorphisms and subrings are unital. If A is a ring, then $\text{tq}(A)$ denotes the total quotient ring

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of A ; $\text{Spec}(A)$ the set of all prime ideals of A ; $\text{Max}(A)$ the set of all maximal ideals of A ; and $\text{Min}(A)$ the set of all minimal prime ideals of A . By an *overring* of A , we mean any ring B such that $A \subseteq B \subseteq \text{tq}(A)$.

If R is a proper subring of a ring T , then $R \subset T$ is called a *minimal ring extension* if the inclusion map $R \hookrightarrow T$ is a minimal ring homomorphism in the sense of [11], that is, if there is no ring S such that $R \subset S \subset T$. (As usual, \subset denotes proper inclusion.) By a *minimal overring* of R , we mean any overring of R which is a minimal ring extension of R . A first step toward the classification of minimal ring extensions was taken by Ferrand–Olivier, who determined the minimal ring extensions of a field in [11, Lemme 1.2], repeated below for convenience as Lemma 2.9. Recently, the authors completed the classification of the minimal ring extensions of a (commutative integral) domain by showing in [10, Theorem 2.7] that if R is a domain but not a field, then the minimal ring extensions of R are the R -algebras that are isomorphic to one of the following three types of rings: a minimal overring of R ; an idealization $R(+R/M$ where $M \in \text{Max}(R)$; a direct product $R \times R/M$ where $M \in \text{Max}(R)$. (For background on the idealization construction, a convenient reference is [16].) This result is generalized in Corollary 2.5, where we obtain the same conclusion assuming only that $\text{tq}(R)$ is a von Neumann regular ring and $\text{Max}(R) \cap \text{Min}(R) = \emptyset$. Examples 2.6–2.7 and Remark 2.8 explore the difficulties in attempting a classification of minimal ring extensions when the base ring R has a maximal ideal which is also a minimal prime ideal (for instance, when R itself is von Neumann regular). Note that any ring R satisfying the hypotheses of Corollary 2.5 must be a reduced ring (that is, 0 is the only nilpotent element of R). Theorem 2.1 and Remark 2.2(a) explain a sense in which the theory of minimal ring extensions reduces to a study of reduced rings.

As possibly suggested above, we obtain our most complete results for the (necessarily reduced) base rings with von Neumann regular total quotient rings (Theorems 2.1 and 3.1 being notable exceptions). However, no such assumption is needed in Proposition 2.12, where we give a generator-and-relation classification of the integral extensions which are minimal ring extensions. Proposition 2.12 is significant in part because whenever $R \subset T$ is a minimal ring extension, either T is integral over R or R is integrally closed in T . Like much of the other work in Section 2, Proposition 2.12 builds on results from [11].

Section 3 addresses the case not covered in the earlier section, namely, the minimal ring extensions $R \subset T$ such that R is integrally closed in T . If R is quasilocal, these extensions are characterized (without any additional hypothesis on $\text{tq}(R)$) in Theorem 3.1. It is shown that any such T must be R -algebra isomorphic to a minimal overring of R . This continues a theme that was implicit in the above rendering of [10, Theorem 2.7], namely, that minimal ring extensions often turn out to be isomorphic to overrings of the base ring. In fact, Sato–Sugatani–Yoshida showed in [20] that if R is a domain which is not a field, then any domain which is a minimal ring extension of R must be (R -algebra isomorphic to) an overring of R . (For some recent generalizations of this result to contexts involving nontrivial zero-divisors, see [19, Proposition 3.9] and [8, Theorem 2.2].) The characterization in Theorem 3.1 generalizes a result of Ayache [2, Theorem 1.2] on quasilocal integrally closed domains that are not fields. As in [2], the characterization in Theorem 3.1 (as well as that in Theorem 3.7) depends on the notion of a divided prime ideal, in the sense of [3,4]. It also depends on the notion of the Kaplansky transform of a maximal ideal. This well-known tool in studying domains was generalized recently to arbitrary rings in [21], from which we recall appropriate material in Section 3. Among the applications of Theorem 3.1 is Proposition 3.11, which generalizes a result recently announced by Picavet and Picavet-L’Hermitte [19, Proposition 5.7] on the minimal overrings of an integrally closed local Noetherian ring.

Most of Section 3 is devoted to obtaining Theorem 3.7. This result generalizes [2, Theorem 2.4], where Ayache used the “divided prime” concept and the Kaplansky transform to characterize the minimal overrings T of any integrally closed domain R which is not a field. Theorem 3.7 generalizes this result in two ways: by assuming only that $\text{tq}(R)$ is von Neumann regular; and by characterizing the minimal ring extensions T of R in which R is integrally closed. Insofar as possible, our path to Theorem 3.7 is patterned after the approach in [2], although we must once again use the generalized Kaplansky transform from [21]. As was the case in Theorem 3.1, it turns out in Theorem 3.7 that any such T must be an overring of R and, up to R -algebra isomorphism, the Kaplansky transform of a uniquely determined maximal ideal of R .

In addition to the notation and terminology introduced above, we use standard notation for various conductors. For instance, given rings $A \subseteq B$, then the conductor $(A : B) := \{b \in B \mid bB \subseteq A\}$. Also, “dimension(al)” refers to Krull dimension; a *regular* element of a ring R is the same as a nonzero-divisor of R ; if $u \in R$, then R_u denotes the localization of R at the multiplicatively closed set generated by u ; and if E is an R -module and $P \in \text{Spec}(R)$, then $E_P := E_{R \setminus P}$. Any unexplained material is standard, as in [14, 16, 17].

2. Reducing to reduced rings, the basic classification, and the integral case

If A is a ring, then we let \sqrt{A} denote the nilradical of A and $A_{\text{red}} := A/\sqrt{A}$ the associated reduced ring of A . If $A \subseteq B$ are rings, then the canonical ring homomorphism $A_{\text{red}} \rightarrow B_{\text{red}}$ is an injection by means of which we view A_{red} as a subring of B_{red} . We begin with a result that explains a sense in which the study of minimal ring extensions reduces to considering extensions of reduced rings.

Theorem 2.1. *Let $R \subset T$ be rings. Then T is a minimal ring extension of R if and only if one of the following two conditions hold:*

- (i) $\sqrt{R} = \sqrt{T}$ and $R_{\text{red}} \subset T_{\text{red}}$ is a minimal ring extension;
- (ii) There exists an element $a \in \sqrt{T} \setminus \sqrt{R}$ such that $T = R[a]$, $a^2 \in (R : T)$, and $(R : T) \in \text{Max}(R)$.

Moreover, if (ii) holds, then $\sqrt{R} \neq \sqrt{T}$, $R_{\text{red}} = T_{\text{red}}$, and $(R :_R a) = (R : T)$.

Proof. Suppose first that $\sqrt{R} = \sqrt{T}$. Then $R_{\text{red}} := R/\sqrt{R} \subset T/\sqrt{R} = T_{\text{red}}$; and it follows from a standard homomorphism theorem that $R \subset T$ is a minimal ring extension if and only if $R_{\text{red}} \subset T_{\text{red}}$ is a minimal ring extension.

Suppose next that $\sqrt{R} \neq \sqrt{T}$ and that T is a minimal ring extension of R . Pick $a \in \sqrt{T} \setminus \sqrt{R}$. As $R \subset R[a] \subseteq T$, it follows from the minimality of $R \subset T$ that $R[a] = T$. Hence, T is integral over R , since $a^n = 0 \in R$ for some integer $n > 1$. By the characterization of integral minimal ring extensions in [11, Proposition 4.1], it follows that there exists $M \in \text{Max}(R)$ such that $MT = M$ and $R/M \subset T/M$ is a minimal ring extension. Since $M \subseteq (R : T) \subset R$, we have that $(R : T) = M$. To prove that (ii) holds, it suffices to show that we can choose a such that $a^2 \in (R : T)$.

Since $a^n = 0 \in (R : T)$, there exists a minimal non-negative integer k such that $a^k \in (R : T)$. Note that $k \geq 2$ since $a \notin R$. Consider $b := a^{k-1}$. For each positive integer i ,

$$ba^i = a^{k+i-1} = a^k a^{i-1} \in (R : T)T \subseteq R.$$

As the minimality of k ensures that $b \notin (R : R[a])$, it follows that $b \notin R$. Therefore, the minimality of $R \subset T$ ensures that $R[b] = T$; and

$$b^2 = a^{2k-2} = a^{k-2}a^k \in T(R : T) = (R : T).$$

As $b \in \sqrt{T} \setminus \sqrt{R}$, the proof of (ii) is complete.

It remains to show that if (ii) holds, then $R \subset T$ is a minimal ring extension such that $\sqrt{R} \neq \sqrt{T}$, $R_{\text{red}} = T_{\text{red}}$, and $(R :_R a) = (R : T)$. It was shown in [13, Corollary 4.13(1)] that if $A \subset B$ are rings and $B = A + Au$ for some element $u \in B$, then $A \subset B$ is a minimal ring extension if and only if $(A : B) \in \text{Max}(A)$. As (ii) ensures that $T = R + Ra$, we can now conclude that T is a minimal ring extension of R . Moreover, $\sqrt{R} \neq \sqrt{T}$ since $a \in \sqrt{T} \setminus \sqrt{R}$. As the minimality of $R \subset T$ yields that $R + \sqrt{T} = T$, the canonical isomorphism

$$T_{\text{red}} = T/\sqrt{T} = (R + \sqrt{T})/\sqrt{T} \cong R/(R \cap \sqrt{T}) = R/\sqrt{R} = R_{\text{red}}$$

yields that $R_{\text{red}} = T_{\text{red}}$. Finally, since $T = R + Ra$, we have that $(R : T) = (R :_R R) \cap (R :_R Ra) = R \cap (R :_R a) = (R :_R a)$. This conclusion may also be shown as follows. If c is any element of $T \setminus R$, then in view of the maximality of M , we see from

$$M = (R : T) \subseteq (R :_R c) \subset R$$

that $(R :_R c) = (R : T)$, to complete the proof. \square

Remark 2.2. (a) The situation described in condition (ii) of Theorem 2.1 is rather well understood. Indeed, let $R \subset T$ be rings. The requirement that $T = R[a]$ for some element $a \in T$ such that $a^2 \in (R : T)$ (which is part of condition (ii) in Theorem 2.1) implies that $T = R + Rt$ for some $t \in T$ (namely, $t = a$). Ring extensions $R \subset T$ satisfying the latter condition were studied by M.S. Gilbert, who showed in [13, Proposition 4.12] that any such data lead to an isomorphism $R/(R : T) \rightarrow T/R$ of R -modules, given by $r + (R : T) \mapsto rt + R$ for all $r \in R$; that each R -submodule contained between R and T is a ring; and that one thereby infers an order-isomorphism between the set of rings intermediate between R and T and the set of ideals of R which contain $(R : T)$.

(b) A ring extension of the form $R \subset T = R[a]$ satisfies $a^2 \in (R : T)$ if and only if $a^k \in R$ for all integers $k \geq 2$. In view of condition (ii) in Theorem 2.1, it is natural to ask if extensions satisfying these conditions must automatically satisfy that $(R :_R a) \in \text{Max}(R)$; or equivalently, to ask whether we can delete the condition that $(R : T) \in \text{Max}(R)$ in condition (ii) in Theorem 2.1. The answer is in the negative. Perhaps the easiest example is provided by taking $R := \mathbb{Z}$ and letting T be the idealization $R(+)R/4R$. Since $R/4R$ is not a simple R -module, it follows from [7, Theorem 2.4] that T is not a minimal ring extension of R . However, $a := (0, 1 + 4R) \in T$ is such that $T = R[a]$ and $a^2 = 0 \in (R : T)$.

(c) Consider a (necessarily minimal) ring extension $R \subset T$ satisfying condition (ii) in Theorem 2.1. The existence of the nonzero nilpotent element a ensures that T is not a reduced ring. Accordingly, it follows from [10, Lemma 2.1] that if R is a domain, then T is R -algebra isomorphic to an idealization $R(+)E$ for some (necessarily simple) R -module E . However, the classification of minimal ring extensions of a non-domain is more complicated, as the preceding assertion fails in general if R is not a domain. To see this, consider the following example. Let S be the ring $\mathbb{Z}/4\mathbb{Z}$ and let M be the unique (up to isomorphism) simple S -module; that is,

$M := 2S = 2\mathbb{Z}/4\mathbb{Z}$. Let R be the idealization $S(+)M$. Then the idealization $T := S(+)S$ is a ring extension of R . Observe that $R \neq T$. Moreover, $a := (0, 1) \in T$ is such that $T = R[a]$ and $a^2 = 0 \in (R : T)$; and $(R : T) = (R :_R a) = M(+)M$, which is the unique maximal ideal of R . Therefore, $R \subset T$ satisfies condition (ii) in Theorem 2.1 (and hence is a minimal ring extension). However, T is not R -algebra isomorphic to an idealization $R(+)E$ for some (necessarily simple) R -module E . In other words, T is not R -algebra isomorphic to $R(+) [R/(M(+)M)]$. Indeed, in view of the uniqueness part of the Fundamental Theorem of Abelian Groups, the underlying additive structures cannot support such an isomorphism, for $T = S \oplus S \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ while $R(+) [R/(M(+)M)]$ is additively isomorphic to

$$(S \oplus M) \oplus (S \oplus M) / (M \oplus M) \cong \mathbb{Z}/4\mathbb{Z} \oplus 2\mathbb{Z}/4\mathbb{Z} \oplus S/M \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

We next note that the pathology exhibited by the example in Remark 2.2(c) can be avoided by assuming that the base ring is reduced.

Proposition 2.3. *Let R be a reduced ring and let T be a minimal ring extension of R . If T is not a reduced ring, then T is R -algebra isomorphic to the idealization $R(+)E$ for some simple R -module E .*

Proof. The proof of [10, Lemma 2.1], which was stated for the case of a domain R , carries over *verbatim*. \square

To go further in studying the minimal ring extensions of a reduced ring R , we impose the condition that $\text{tq}(R)$ is a von Neumann regular ring. Indeed, if $\text{tq}(R)$ is a von Neumann regular ring and $\text{Max}(R) \cap \text{Min}(R) = \emptyset$, then Corollary 2.5 classifies the minimal ring extensions of R . Combining this result with the classification of the minimal ring extensions of a field in [11, Lemma 1.2], we obtain, as a special case, a classification of the minimal ring extensions of a domain, thus generalizing the main result of [10]. Examples 2.6 and 2.7 explain why the condition “ $\text{Max}(R) \cap \text{Min}(R) = \emptyset$ ” in Corollary 2.5 is indispensable. Most of the work in proving Corollary 2.5 is accomplished in Theorem 2.4. Prior to that result, we devote the next paragraph to some additional background material.

Let $R \subset T$ be a minimal ring extension. By [11, Théorème 2.2(i) and Lemme 1.3], there exists a unique maximal ideal M of R such that $R_M \hookrightarrow T_M := T_{R \setminus M}$ is not an isomorphism; moreover, $R_M \hookrightarrow T_M$ is then a minimal ring extension, and $R_P \hookrightarrow T_P$ is an isomorphism for all $P \in \text{Spec}(R) \setminus \{M\}$. Following [5], we call M the *crucial maximal ideal* of the minimal extension $R \subset T$. According to [11, Théorème 2.2(iii)], either T is integral over R or $R \hookrightarrow T$ is a flat epimorphism. If M is the crucial maximal ideal of $R \subset T$, then [11, Théorème 2.2(ii)] yields that T is integral over R if and only if $MT = M$, that is, if and only if $(R : T) = M$. If $Q := \text{tq}(R)$ is a von Neumann regular ring, it is known that each flat epimorphic ring extension S of R is (R -algebra isomorphic to) an overring of R . (A proof can be found in [15, Theorem 4.3.7]. The referee has kindly provided the following short proof. The canonical R -algebra homomorphism $f : Q \rightarrow Q \otimes_R S$ is injective since Q is R -flat; and f is a ring epimorphism since $R \hookrightarrow S$ is a ring epimorphism. Thus, f is surjective since Q , being locally a field, cannot be the source of a ring epimorphism that is not an isomorphism. Accordingly, as S is R -flat, S can be identified, as an R -algebra, with a subring of $Q \otimes_R S \cong Q$.) Thus, if $\text{tq}(R)$ is von Neumann regular and T is not (R -algebra isomorphic to) an overring of R , then T is integral over R and the crucial maximal ideal M of $R \subset T$ satisfies $M = MT = (R : T)$. Finally, if R is *any* nonzero ring

and $M \in \text{Max}(R)$, then it follows from case (b) of [11, Lemme 1.5] that the canonical ring homomorphism $R \hookrightarrow R \times R/M$ is a minimal ring extension; and it is then straightforward to verify that M is the crucial maximal ideal of $R \subset R \times R/M$. In particular, each maximal ideal of a ring R is the crucial maximal ideal of some minimal ring extension of R . Note that the same conclusion follows from [7, Corollary 2.5] if one replaces $R \times R/M$ with the minimal ring extension $R(+R/M)$ of R .

Theorem 2.4. *Let R be a (necessarily reduced) ring such that Q , the total quotient ring of R , is a von Neumann regular ring. Let T be a minimal ring extension of R , and let M denote the crucial maximal ideal of $R \subset T$. Suppose that M is not a minimal prime ideal of R . (Thus $R \neq Q$.) Then T is R -algebra isomorphic to either the product $R \times R/M$, the idealization $R(+R/M)$, or an overring of R .*

Proof. The first parenthetical assertion follows since von Neumann regular rings are reduced. The second parenthetical assertion follows since M has nonzero height in R (as $M \notin \text{Min}(R)$) while Q , being von Neumann regular, must be zero-dimensional. We begin with a couple of reductions. First, in view of Proposition 2.3 and the final comment preceding the statement of the present result, we may suppose that T is a reduced ring. Next, without loss of generality, we can suppose that T is not R -algebra isomorphic to an overring of R . It remains to show that $R \cong R \times R/M$.

We claim that M contains a regular element of R . As $M \notin \text{Min}(R)$, there exists $P \in \text{Min}(R)$ such that $P \subset M$ [17, Theorem 10]; and $P \subset Z(R)$ by the case $A := R$ of [17, Theorem 84]. As $Q = R_{R \setminus Z(R)}$, it follows that PQ is a (necessarily minimal) prime ideal of Q . However, PQ is also a maximal ideal of Q , since Q is zero-dimensional. As $PQ \subset MQ$, we see that $MQ = Q$. It follows that $1 \in MR_{R \setminus Z(R)}$, whence M contains an element of $R \setminus Z(R)$, and the above claim has been proved.

By the above claim, we can choose $b \in M$ which is a regular element of R . By the above comments about flat epimorphic extensions, we have that $(R : T) = M$, and so $bT \subseteq R$. It cannot be the case that b is a regular element of T , as T is not isomorphic to an overring of R . (In detail, if $b \notin Z(T)$, then the calculation $t = (bt)/b$ shows that the canonical ring homomorphism $T \rightarrow T_b$ is an injection whose image lies in (the canonical image of) $R_b \subseteq Q$.) As $b \in Z(T)$, there exists a nonzero element $c \in T$ such that $bc = 0$. On the other hand, $Mc \subseteq (R : T)T \subseteq R$. Since b is a regular element of R , the equation $b(Mc) = (bc)M = 0M = 0$ yields that $Mc = 0$. In particular, M is not a faithful ideal of T . As it follows from the above comments that T is integral over R , an appeal to [11, Proposition 4.6] completes the proof. \square

We can now present the “basic classification” referred to in the title of this section.

Corollary 2.5. *Let R be a ring such that Q , the total quotient ring of R , is von Neumann regular and no maximal ideal of R is a minimal prime ideal of R . Then the minimal ring extensions of R are, up to R -algebra isomorphism, of the following three types:*

- (i) *Minimal overrings of R , that is, R -subalgebras of Q that are minimal ring extensions of R ;*
- (ii) *For each maximal ideal M of R , the idealization $R(+R/M)$;*
- (iii) *For each maximal ideal M of R , the ring $R \times R/M$.*

Moreover, the above listing is a classification, in the following sense. The above types (i)–(iii) of R -algebras do not overlap; if M and N are distinct maximal ideals of R , then $R(+)R/M$ is not R -algebra isomorphic to $R(+)R/N$ and $R \times R/M$ is not R -algebra isomorphic to $R \times R/N$; and if A and B are isomorphic R -subalgebras of Q , then $A = B$.

Proof. For any ring R , any R -algebra T of type (i), (ii) or (iii) is a minimal ring extension of R . Indeed, there is nothing to prove in case (i); [7, Corollary 2.5] takes care of any T satisfying (ii); and, as explained in the paragraph prior to Theorem 2.4, [11, Lemme 1.5] takes care of any T satisfying (iii). Conversely, in view of the prevailing assumptions on R , Theorem 2.4 shows that any minimal ring extension T of R must be of one of the types (i)–(iii), since the crucial maximal ideal of $R \subset T$ cannot be a minimal prime ideal of R .

It remains to address issues involving uniqueness in the above classification. Here, the possible minimality of the R -algebras plays no role. We first address the assertion that the types “do not overlap.” To show that no R -algebra of type (i) can be of type (ii), it suffices to apply [7, Remark 2.8], where it was shown that if R is any ring and E any nonzero R -module, then $R(+)E$ is not R -algebra isomorphic to an overring of R . In addition, since R is a reduced ring, each R -algebra of type (iii) is reduced and hence cannot also be of type (ii) (and indeed cannot be R -algebra isomorphic to $R(+)E$ for some nonzero R -module E). Lastly, to show that types (i) and (iii) do not overlap, we suppose, on the contrary, that there exists $M \in \text{Max}(R)$ such that $T := R \times R/M$ is R -algebra isomorphic to an overring of R . Since $T \setminus Z(T) = \{(a, \alpha) \in T \mid a \in R \setminus Z(R) \text{ and } \alpha \neq 0\}$, one sees easily that there is an R -algebra isomorphism $\text{tq}(T) \cong Q \times R/M$, and so there is an R -algebra isomorphism $f: Q \rightarrow Q \times R/M$. Now, if $r \in R$ and $z \in R \setminus Z(R)$, then

$$zf\left(\frac{r}{z}\right) = f\left(z\frac{r}{z}\right) = f(r) = r(1, \bar{1}) = (r, \bar{r}),$$

where $\bar{b} := b + M \in R/M$ for each $b \in R$. It follows that

$$f\left(\frac{r}{z}\right) = (r, \bar{r})(z, \bar{z})^{-1} = (r, \bar{r})\left(\frac{1}{z}, \bar{z}^{-1}\right) = \left(\frac{r}{z}, \frac{\bar{r}}{\bar{z}}\right).$$

Therefore, the element $(0, \bar{1}) \in Q \times R/M$ is not in the image of f , the desired contradiction, thus proving that types (i) and (iii) do not overlap.

Finally, we show that the above listing has no overlaps within types. Type (ii) is easiest to address, for it follows from [7, Lemma 2.6] that if R is any ring and I and J are incomparable prime ideals of R , then $R(+)R/I$ is not R -algebra isomorphic to $R(+)R/J$. An easier proof follows from the fact that if I is an ideal of a ring R , then the conductor $(R : R(+)R/I) = I$. This alternate argument suggests how to dispatch possible overlaps within (iii), for one checks easily that if I is an ideal of a ring R , then the conductor $(R : R \times R/I) = I$. It remains only to address possible overlaps within (i), that is, to show that if R is a ring with $Q = \text{tq}(R)$ and A and B are isomorphic R -subalgebras of Q , then $A = B$. With minor changes, this can be handled as in the case of a domain R : see the first paragraph of [10, Remark 2.8(a)]. The proof is complete. \square

The next two results show that one cannot eliminate the hypothesis that $\text{Max}(R) \cap \text{Min}(R) = \emptyset$ from Corollary 2.5. The ring R in Example 2.6 (respectively, 2.7) is a von Neumann regular ring with only finitely many (respectively, infinitely many) prime ideals.

Example 2.6. Let $n \geq 2$ be an integer. Then there exist a von Neumann regular ring R with exactly n prime (i.e., maximal) ideals and a minimal ring extension T of R such that T is not R -algebra isomorphic to any ring of type (i), (ii), or (iii) (using the notation in the statement of Corollary 2.5). It can also be arranged that T is a von Neumann regular ring R with exactly n prime ideals.

Proof. Consider any minimal field extension $K \subset L$, and let $K = K_1, \dots, K_n$ be a list of fields. Then $R := K_1 \times K_2 \times \dots \times K_n$ is a von Neumann regular ring R with exactly n prime ideals, and so is $T := L \times K_2 \times \dots \times K_n$. Moreover, T is a minimal ring extension of R . Since $\text{tq}(R) = R \subset T$, we see that T is not of type (i). In addition, since T is a reduced ring, T is not of type (ii). Finally, T is not of type (iii), for if $M \in \text{Max}(R)$, then any ring that is isomorphic to $R \times R/M$ must have $n + 1$ prime ideals. \square

Example 2.7. There exist a von Neumann regular ring R with infinitely many prime (i.e., maximal) ideals and a minimal ring extension T of R such that T is not R -algebra isomorphic to any ring of type (i), (ii), or (iii) (using the notation in the statement of Corollary 2.5). If K is a field which is not algebraically closed, L is a minimal field extension of K , and $K = K_1, \dots, K_n, \dots$ is a denumerable list of fields each of which contains K and none of which is isomorphic to L , then suitable R, T may be constructed as follows. Take R to be the K -subalgebra of $\prod_{i=1}^{\infty} K_i$ which is generated by $\bigoplus K_i$, that is, R consists of the sequences $(r_i)_{i \geq 1}$ such that $r_i \in K_i$ for each $i \geq 1$ and $r_n = r_{n+1} = \dots \in K$ for some n ; and obtain T by letting L play the role of K_1 in the construction of R , that is, T consists of the sequences $(t_i)_{i \geq 1}$ such that $t_1 \in L$, $t_i \in K_i$ for each $i \geq 2$ and $t_m = t_{m+1} = \dots \in K$ for some m . This construction also arranges that T is a von Neumann regular ring R with infinitely many prime ideals.

Proof. An easy calculation (using the “ $aba = a$ ” criterion) shows that R and T are each von Neumann regular rings. Since the i th projection map $R \rightarrow K_i$ is surjective, its kernel, denoted by M_i , must be a maximal ideal of R . If j and k are distinct positive integers, consider any sequence $s \in R$ whose only nonzero entry is in the k th coordinate. Evidently, $s \in M_j \setminus M_k$, and so $M_j \neq M_k$. Thus, R has infinitely many prime ideals. Similarly, so does T .

Next, we show that $R \subset T$ is a minimal ring extension. It is enough to show that $R[t] = T$ for each element $t = (t_i) \in T \setminus R$. If $b = (b_i) \in T$, then $(0, b_2, b_3, \dots) \in R \subset R[t]$. It follows that $(t_1, 0, 0, \dots) \in R[t]$. Also, since $K \subset L$ is a minimal field extension, $K[t_1] = L$. Hence, $(c, 0, 0, \dots) \in R[t]$ for each $c \in L$. Thus, $b = (0, b_2, b_3, \dots) + (b_1, 0, 0, \dots) \in R[t]$, whence $R[t] = T$, as desired.

It remains to show that T is not isomorphic to an R -algebra of type (i), (ii), or (iii). Type (i) is dispatched since $\text{tq}(R) = R \subset T$, while T cannot be of type (ii) since T is a reduced ring. To complete the proof, we will show that there does not exist $M \in \text{Max}(R)$ such that T is R -algebra isomorphic to $R \times R/M$.

Let N_1 denote the kernel of the (first) projection map $T \rightarrow L$; then $T/N_1 \cong L$ as R -algebras. If the assertion fails, $T \cong R \times R/M$ for some $M \in \text{Max}(R)$. Thus, by the Second Isomorphism Theorem for R -algebras, T/N_1 is isomorphic to some factor ring of $R \times R/M$. It follows that $L \cong R/P$ as R -algebras for some $P \in \text{Spec}(R)$. Hence, there is a surjective R -algebra homomorphism $R \rightarrow L$. However, there is only one R -algebra map $R \rightarrow L$, namely, the composite of the inclusion map $R \hookrightarrow T$ and the projection map $T \rightarrow L$. The image of this map is $K \neq L$, the desired contradiction. \square

Remark 2.8. (a) The construction in Example 2.7 could be modified to become more analogous to that in Example 2.6. Indeed, the above proof also applies if one redefines R as $\prod_{i=1}^{\infty} K_i$ and T as $L \times \prod_{i=2}^{\infty} K_i$. The construction using eventually constant sequences was used in Example 2.7 because it produces somewhat smaller rings. For instance, if each K_i is a finite field, then (so is L and) the ring R in Example 2.7 is denumerable whereas the cardinality of $\prod_{i=1}^{\infty} K_i$ is then 2^{\aleph_0} .

(b) Examples 2.6 and 2.7 illustrate the difficulties in extending Corollary 2.5 to a classification result for the minimal ring extensions of a (necessarily reduced) ring R such that $\text{tq}(R)$ is von Neumann regular in case R has more than one maximal ideal and $\text{Max}(R) \cap \text{Min}(R)$ is nonempty. However, such a classification is at hand if R is quasilocal. Indeed, if such R is (quasilocal and) zero-dimensional, it must be a field and so [11, Lemme 1.2] accomplishes the desired classification. On the other hand, if a quasilocal ring R has nonzero dimension and a von Neumann regular total quotient ring, then $\text{Max}(R) \cap \text{Min}(R) = \emptyset$ and so Corollary 2.5 can be applied.

(c) In view of Examples 2.6 and 2.7, it would be of interest to know the answer to the following special case of the question mentioned in (b): if R is a von Neumann regular ring, can one classify the R -algebra isomorphism classes of the minimal ring extensions of R ? Since R is von Neumann regular, $\text{tq}(R) = R$, and so (using the notation from Corollary 2.5) there is no R -algebra of type (i). Our earlier reasoning shows that any R -algebra of type (ii) or (iii) is a minimal ring extension of R ; and that the uniqueness assertions in Corollary 2.5 apply in general (to any von Neumann regular ring R). Thus, the question comes down to this: if T is a minimal ring extension of a von Neumann regular ring R such that T is not of the types (i), (ii), or (iii), what can be said about T ? We next offer three conclusions (but leave the question open). First, by Proposition 2.3, any such T must be a reduced ring. Second, by the results that were recalled prior to the statement of Theorem 2.4, T must be integral over R and so the conductor $(R : T)$ coincides with the crucial maximal ideal of $R \subset T$. Third, if (the von Neumann regular ring) R has only finitely many prime (maximal) ideals (i.e., if R is a direct product $R = K_1 \times \cdots \times K_n$ for some finite list of fields K_i), then the minimal ring extensions of R are the R -algebras that are isomorphic to direct products of the form $T_1 \times \cdots \times T_n$ where there exists $j \in \{1, \dots, n\}$ such that $T_i = K_i$ if $i \neq j$ and T_j is a minimal ring extension of K_j .

In view of Corollary 2.5 and Remark 2.8(c), it seems appropriate to study more deeply the integral minimal overrings of a given ring R . More generally, we devote the rest of this section to characterizing the integral minimal ring extensions of any ring R . (Like much of the earlier part of Section 2, the characterization of the non-integral minimal overrings of R in Section 3 proceeds under the hypothesis that $\text{tq}(R)$ is von Neumann regular.) Recall from [11, Proposition 4.1] that a ring extension $A \subset B$ is an integral minimal ring extension if and only if some $M \in \text{Max}(A)$ is such that $MB = B$ (i.e., $(A : B) = M$) and B/M is a minimal ring extension of the field A/M . (When these conditions hold, it was noted in [5, Remark 2.10] that M must be the crucial maximal ideal of $A \subset B$.) This insight leads to a more detailed characterization (via generators and relations) of the integral minimal overrings of R , as an application of the tripartite classification in Proposition 2.12. First, we recall (for reference purposes) the classification of the minimal ring extensions of a field.

Lemma 2.9. (Ferrand–Olivier [11, Lemme 1.2]) *Let L be a ring extension of a field K . Then L is a minimal ring extension of K if and only if (exactly) one of the following three conditions holds:*

- (1) L is a minimal field extension of K ;
- (2) $L \cong K \times K$ as K -algebras (where the embedding $K \rightarrow K \times K$ is via the diagonal map);
- (3) $L \cong K[X]/(X^2)$, the ring of dual numbers over K , as K -algebras.

Example 2.10 collects examples of integral minimal overring extensions $R \subset T$, with crucial maximal ideal M , such that T/M is the second or third type of R/M -algebra listed in Lemma 2.9.

Example 2.10. Let K be any field and consider the polynomial ring $T = K[X]$. Then:

- (a) Let $R = K[\{X^n(X-1) \mid n = 1, 2, \dots\}]$. Then T is an integral minimal overring of R , with $M := (R : T) = (\{X^n(X-1) \mid n = 1, 2, \dots\}) = (X(X-1), X^2(X-1))$. Furthermore, $T/M \cong R/M \times R/M$ as R/M -algebras.
- (b) Let $R = K[X^2, X^3]$. Then T is an integral minimal overring of R , with $M := (R : T) = (X^2, X^3)$. Furthermore, $T/M \cong (R/M)[X]/(X^2)$.

Proof (Sketch). One proof is immediate by applying Proposition 2.12 (with $q := X$). A direct proof is also possible; we leave its details to the reader. \square

Proposition 2.12 will establish that the constructions in Example 2.10(a), (b) are typical of two of the three classes of integral minimal overring extensions $R \subset T$. In the spirit of [10, Theorem 2.3(c) and Remark 2.5], we show that linear and quadratic relations suffice to describe such T . First, we isolate a result of some independent interest.

Lemma 2.11. Let $R \subset T = R[x]$ be a ring extension, where x is an element of T which is a root of some monic polynomial of degree 2 in $R[X]$. If there exists a maximal ideal M of R such that $Mx \subseteq R$, then T is a minimal ring extension of R .

Proof. It suffices to prove that $R[y] = T$ for each $y \in T \setminus R$. The hypotheses on x ensure that $T = R + Rx$, and so $y = ax + b$ for some $a, b \in R$. Since $R[y - b] = R[y]$, we may suppose (by replacing y with $y - b$) that $b = 0$. As $ax = y \notin R$ and $Mx \subseteq R$, we have that $a \notin M$. Then the maximality of M yields $M + Ra = R$, and so $m + sa = 1$ for some $m \in M$ and $s \in R$. Hence,

$$x = (m + sa)x = mx + s(ax) \in Mx + R[ax] \subseteq R + R[y] = R[y] \subseteq R[x],$$

and so $R[y] = R[x] = T$, as desired. \square

Proposition 2.12. Let R be a ring with total quotient ring Q . Let T be an integral ring extension of R . (Hence, if $R \neq Q$, then $T \not\supseteq Q$.) Then T is a minimal ring extension of R if and only if there exists $M \in \text{Max}(R)$ such that one of the following three conditions holds:

- (1) M is a maximal ideal of T and T/M is a minimal field extension of R/M ;
- (2) There exists $q \in T \setminus R$ such that $T = R[q]$, $q^2 - q \in M$, and $Mq \subseteq R$;
- (3) There exists $q \in T \setminus R$ such that $T = R[q]$, $q^2 \in R$, $q^3 \in R$, and $Mq \subseteq R$.

If any of the above three conditions holds, then M is uniquely determined as $(R : T)$, the crucial maximal ideal of the extension $R \subset T$. Furthermore, conditions (1)–(3) are mutually exclusive.

Indeed, if T is an integral minimal ring extension of R , then (2) (respectively, (3)) is equivalent to T/M being isomorphic as an R/M -algebra to $R/M \times R/M$ (respectively, $(R/M)[X]/(X^2)$).

Proof. The parenthetical assertion follows because the only integral overring of R which is R -flat is R itself. Now, assume that T is a minimal ring extension of R . By the material recalled prior to Lemma 2.9, the conductor $(R : T) =: M$ is the crucial maximal ideal of $R \subset T$ and T/M is a minimal ring extension of the field R/M . Thus, by Lemma 2.9, there are three mutually exclusive possibilities: either T/M is a field (in which case, (1) holds), T/M can be identified with $R/M \times R/M$ as an R/M -algebra, or T/M can be identified with $(R/M)[X]/(X^2)$ as an R/M -algebra.

If $T/M = R/M \times R/M$, choose $f = f^2 \in T/M \setminus R/M$ (for instance, take $f = (0, 1 + M)$) and then choose $q \in T$ such that $f = q + M$. As $q \notin R$, we have $R \subset R[q] \subseteq T$, and so $R[q] = T$ by the minimality of T . Moreover, $q^2 - q \in M$ since $f^2 - f = 0$; and $Mq \subseteq (R : T)T \subseteq R$. Therefore, condition (2) holds if $T/M \cong R/M \times R/M$.

In the remaining case, we can identify $T/M = (R/M)[X]/(X^2)$, and we will show that condition (3) holds. Indeed, if we choose a nonzero element $h \in T/M$ such that $h^2 = 0$ (for instance, take $h = X + (X^2)$) and then choose $q \in T$ such that $h = q + M$, it is easy to see that (3) is satisfied. This completes the proof of the “only if” assertion.

For the converse, we will show that each of (1)–(3) implies that T is a(n integral) minimal ring extension of R . If (1) holds, this is a consequence of the Second Isomorphism Theorem (cf. [9, Lemma II.3]). On the other hand, if either (2) or (3) holds, this is a consequence of Lemma 2.11. This completes the proof of the “if” assertion.

If any of (1), (2) or (3) holds, note that $M \subseteq (R : T) \subset R$, whence M is uniquely determined as $(R : T)$. It remains to show that the conditions (1)–(3) are mutually exclusive, with (2) and (3) corresponding to T/M being isomorphic to $R/M \times R/M$ and $(R/M)[X]/(X^2)$, respectively. We already know, from the proof of the “only if” assertion, that (2) (respectively, (3)) is implied by T/M being isomorphic to $R/M \times R/M$ (respectively, $(R/M)[X]/(X^2)$).

Suppose that (2) holds. Then $q + M$ is an idempotent element which generates T/M as an R/M -algebra. As $T/M \neq R/M$, it follows that $q + M$ cannot be 0 or 1. In particular, T/M cannot be a field; that is, (1) cannot hold. In addition, T/M cannot be isomorphic to $(R/M)[X]/(X^2)$, since 0 and 1 are the only idempotent elements in the ring of dual numbers over any field. It follows, by the process of elimination, that $T/M \cong R/M \times R/M$.

Suppose next that (3) holds. As $q \notin R$, we have, *a fortiori*, that $q \notin M$. Consider the nonzero element $y := q + M \in T/M$. If T/M were a field, we would have that the field R/M contains $(q^3 + M)(q^2 + M)^{-1} = y$, whence $q \in R$, a contradiction. Therefore, (1) cannot hold. By the process of elimination, it remains only to show (given (3)) that T/M cannot be isomorphic to $R/M \times R/M$. Since $R/M \times R/M$ is a reduced ring, it suffices to find a nonzero nilpotent element in T/M . As $y \neq 0$, it therefore suffices to show that $q^2 \in M$ (for then $y^2 = 0$). Since $M = (R : T)$ and (3) ensures that $T = R + Rq$, we need only note (since $q^2, q^3 \in R$) that $q^2T \subseteq q^2R + q^3R$. \square

Remark 2.13. (a) In contrast to the R -algebras in cases (2) or (3) of Proposition 2.12, the rings T in case (1) cannot, in general, be presented as R -algebras via linear or quadratic relations. For instance, if $R = \mathbb{Q}$ and $n \geq 2$ is an integer, then there exists a minimal field extension T of R such that $[T : R] \geq n$. Presenting such T as an R -algebra $R[X]/I$, we see that I must have a generator of degree at least n , namely, the minimal polynomial of a primitive element of the field extension $R \subset T$.

(b) Proposition 2.12 may be viewed as a companion for [9, Corollary II.2]. The latter result gives another tripartite classification of the integral minimal ring extensions of a given ring R and, like Proposition 2.12, is also a consequence of [11, Proposition 4.1 and Lemme 1.2]. Unlike the generator-and-relation approach in Proposition 2.12, the formulation of [9, Corollary II.2] is ideal-theoretic.

3. The integrally closed case

In this section, we examine minimal ring extensions $R \subset T$ such that R is integrally closed in T . If R is a domain, the classification in [10, Theorem 2.7] shows that any such T must be R -algebra isomorphic to an overring of R . We will see that the same holds true more generally, namely, if $\text{tq}(R)$ is von Neumann regular. The main results in this section generalize the work of Ayache [2] who showed, among other things, that if R is an integrally closed domain but not a field, then each minimal overring of R is the Kaplansky transform of a maximal ideal that satisfies certain properties. In [21], a generalized notion of the Kaplansky transform was introduced and developed. In the main result of this section, we will apply this notion to the case where $\text{tq}(R)$ is von Neumann regular. With this assumption, if R is integrally closed in the minimal ring extension T of R , then T is a flat epimorphic extension of R [11, Théorème 2.2(iii)] and, as noted earlier, it then follows from [15, Theorem 4.3.7] that T is R -algebra isomorphic to an overring of R . By the minimality of the ring extension, T is clearly finitely generated as an R -algebra. Thus, by [21, Corollary 3.9], T is the (generalized) Kaplansky transform of some ideal of R . We will show in Theorem 3.7 that T is, in fact, the (generalized) Kaplansky transform of a maximal ideal with the same restrictive properties as noted by Ayache, together with a new condition concerning regular elements that is required here because we are venturing beyond domains.

The first result of the section is a generalization of [2, Theorem 1.2] (where R was assumed to be a quasilocal integrally closed domain but not a field). Theorem 3.1 concerns quasilocal base rings and does not need any additional assumptions on $\text{tq}(R)$. However, we reiterate that in order to globalize this result in Theorem 3.7, we will have to assume that $\text{tq}(R)$ is von Neumann regular.

Many of the proofs in this section use the following definition. Recall that a prime ideal P of a ring R is called (a) *divided (prime ideal of R)* if, for each ideal I of R , either $I \subset P$ or $P \subseteq I$.

Theorem 3.1. *Let (R, M) be a quasilocal ring. Then the following conditions are equivalent:*

- (1) *There exists a minimal ring extension T of R such that R is integrally closed in T ;*
- (2) *There exists a divided prime ideal P of R such that R/P is a one-dimensional valuation domain and there exists an element $u \in M \setminus P$ which is a regular element of R .*

Moreover, if the above conditions hold, then each element of $M \setminus P$ is a regular element of R , $\text{Rad}(u) = M$, and $T \cong R_P \cong R_u$ is R -algebra isomorphic to an overring of R . In particular, any two minimal ring extensions of R in which R is integrally closed must be isomorphic as R -algebras.

Proof. (1) \Rightarrow (2): Assume (1). As T is not integral over R , it follows from [11, Théorème 2.2(iii)] that T is a flat epimorphic extension of R . Thus, by [19, Proposition 3.5], if $P := (R : T)$, then R/P is a one-dimensional valuation domain with quotient field T/P . In addition, by [19,

Proposition 3.2(3)], ($P \in \text{Max}(T)$) and $T = R_P$. (In fact, the cited result yields that $T_M = R_P$; note that our assertion then follows because R is quasilocal.) Since the canonical ring homomorphism $R \rightarrow R_P$ is therefore an injection, each element of $R \setminus P$ must be a regular element of R .

Of course, $P \neq M$, since R/P is not a field. By the argument of the preceding paragraph, each element of $M \setminus P$ is a regular element of R . It remains only to show that P is divided. Choose $x \in M \setminus P$. We must show that $p \in Rx$ for each $p \in P$. Suppose not, and pick $p \in P \setminus Rx$. Since $T = R_P$, we have $v := p/x \in T$. Hence, $R[v] = T$, by the minimality of T . Note that $v^2 \notin R$, since R is integrally closed in T . Thus, $R[v^2] = T$, and so $v \in R[v^2]$. Therefore, v is a root of some polynomial in $R[X]$ having at least one of its coefficients being a unit of R . Hence, by Seidenberg's u, u^{-1} Lemma (as in [17, Exercise 31, p. 43]), $v^{-1} \in R$ (since $v \notin R$). Then $(p/x)r = vr = 1 \in T$ for $r := v^{-1} \in R$, whence $x = pr \in P$, a contradiction.

(2) \Rightarrow (1): Let P be as in (2) and let $u \in M \setminus P$ be a regular element of R . We will show that $T := R_u$ is a minimal ring extension of R and that R is integrally closed in T . Of course, the hypothesis on u ensures that the canonical ring homomorphism $R \rightarrow T$ is an injection, and so we can regard T as a ring extension of R .

If n is a positive integer, $u^n \in R \setminus P$ and so, since P is a divided prime ideal, $P \subset Pu^n$. It follows (if we view $R \subseteq T$ as above) that P is an ideal of T . Thus, $R/P \subseteq T/P$. Since u is a unit of $R_u = T$, it follows that $y := u + P \in R/P$ is a unit of T/P , and so $(T/P)_y \cong T/P$ canonically. By applying [12, Proposition 1.9] to the pullback $R = (R/P) \times_{T/P} T$ and the multiplicatively closed set $\{u^n \mid n = 1, 2, \dots\}$ of R , we obtain the pullback $R_u = (R/P)_y \times_{(T/P)_y} T_u$. As $T_u = (R_u)_u \cong R_u = T$ canonically, we obtain the pullback $T = (R/P)_y \times_{T/P} T$. Hence, the canonical ring homomorphism $(R/P)_y \rightarrow T/P$ is an isomorphism.

Now, since y is a nonzero nonunit of the one-dimensional valuation domain R/P , it follows that $(R/P)_y$ is the quotient field of R/P (cf. [14, Theorem 26.1(2)], [17, Exercise 29, p. 43]); that is, T/P is the quotient field of R/P . Since valuation domains are integrally closed, R/P is integrally closed in T/P ; and, by the just-cited references, T/P is a minimal ring extension of R/P . It follows that R is integrally closed in T (cf. [12, Corollary 1.5(5)]); and that T is a minimal ring extension of R (cf. [9, Lemma II.3]).

It remains to show that if (1) and (2) hold, then $\text{Rad}(u) = M$ for each $u \in M \setminus P$; that T is R -algebra isomorphic to an overring of R ; and that, up to R -algebra isomorphism, the quasilocal ring R has only one minimal ring extension in which R is integrally closed. The first assertion is immediate since M is the only prime ideal of R that properly contains any P that satisfies (2). Moreover, the third assertion holds since the dividedness of such P (together with the fact that R/P is a quasilocal one-dimensional ring) proves that P is unique, whence $T (\cong R_P)$ is determined up to R -algebra isomorphism. Finally, we offer two proofs that T is R -algebra isomorphic to an overring of R . For the first of these, recall that each element of $R \setminus P$ is a regular element of R , and so the universal mapping property of rings of quotients gives an R -algebra homomorphism $R_P \rightarrow \text{tq}(R)$ which is easily seen to be an injection. The second proof that T embeds in $\text{tq}(R)$ is an application of [8, Theorem 2.2]: it suffices to note that P is a nonmaximal prime ideal of R that contains all the zero-divisors of R and that every regular element in R remains a regular element in $R_P (= T)$. \square

The following uniqueness result is a consequence of Theorem 3.1.

Corollary 3.2. *Let R be a quasilocal ring and Q a given total ring of quotients of R . Then:*

- (a) *If T is a minimal ring extension of R such that R is integrally closed in T , then there exists a unique ring A such that $R \subset A \subseteq Q$ and $T \cong A$ as R -algebras.*

- (b) *There do not exist distinct R -subalgebras A_1, A_2 of Q such that $R \subset A_i$ is a minimal ring extension and R is integrally closed in A_i for $i = 1, 2$.*

Proof. By Theorem 3.1, it suffices to prove that any isomorphic R -subalgebras of Q must coincide. As noted in the proof of Corollary 2.5, this can be done by modifying an argument in the first paragraph of [10, Remark 2.8(a)]. \square

Additional applications of Theorem 3.1 will be given in Remark 3.9 and Proposition 3.11. We turn now to the global case, by recalling a definition and some results from [21]. If I is an ideal of a ring R , then the collection of ideals $\mathcal{F}_I := \{J \subseteq R \mid I \subseteq \text{Rad}(J)\}$ is a localizing (or Gabriel) filter canonically associated to I . The *generalized Kaplansky transform of R with respect to the ideal I* , denoted by $\Omega_R(I)$ or $\Omega(I)$ if R is understood, is defined to be the ring of quotients of R with respect to \mathcal{F}_I . For details on this construction and related results, see [21, Section 3].

Suppose that I is a regular ideal of a ring R ; that is, I contains a regular element of R . Then each ideal J in \mathcal{F}_I is a regular ideal of R (since J contains a power of a regular element in I). It follows that $\Omega(I)$ is a subring of Q , the total quotient ring of R . In fact, as noted in the discussion prior to [21, Proposition 2.6], $\Omega(I) = \{q \in Q \mid qJ \subseteq R \text{ for some } J \in \mathcal{F}_I\}$. It will be useful to note the equivalent formula $\Omega(I) = \{q \in Q \mid I \subseteq \text{Rad}(R : q)\}$.

If M is a maximal ideal of a ring R , then \mathcal{F}_M is simply the set of ideals $\{J \subset R \mid \text{Rad}(J) = M\} \cup \{R\}$. Assume, in addition, that $Q = \text{tq}(R)$ is von Neumann regular and that (the maximal ideal) M is not a minimal prime ideal of R . Then, as in the second paragraph of the proof of Theorem 2.4, M contains a regular element of R . Therefore, by taking $I = M$ in the preceding paragraph, we have

$$\Omega(M) = \{q \in Q \mid \text{Rad}(R : q) = M\} \cup R;$$

in the spirit of [2], it is useful to note the equivalent formula

$$\Omega(M) = \bigcap_{y \in M} \bigcup_{n=1}^{\infty} (R :_Q y^n).$$

We will show in Lemma 3.3 that, under the above conditions, one has (as in the case of a domain R), that the localizations R_P and $\Omega(M)_P$ coincide for any prime ideal P other than M . It is convenient to first note some terminology. Let $P \in \text{Spec}(R)$ and let $A \subseteq B$ be R -modules. Then we say that B/A is *P -torsion* if $(A :_R b) \not\subseteq P$ for each $b \in B$; and we say that B/A is *P -torsion-free* if $(A :_R b) \subseteq P$ for each $b \in B \setminus A$. It is easy to check that B/A is P -torsion if and only if the canonical injection $A_P \rightarrow B_P$ is an isomorphism (in which case we write $A_P = B_P$); and that B/A is P -torsion-free if and only if the canonical map $B/A \rightarrow (B/A)_P$ is an injection.

Each ring R that is considered henceforth has a von Neumann regular total quotient ring, but for the sake of clarity, we will include this now-riding hypothesis in the statements of results.

Lemma 3.3. *Let R be a ring such that Q , the total quotient ring of R , is von Neumann regular. Let $M \in \text{Max}(R)$ such that M is not a minimal prime ideal of R . (Hence, $\Omega(M) \subseteq Q$.) Then $R_P = \Omega(M)_P$ for each $P \in \text{Spec}(R) \setminus \{M\}$.*

Proof. The parenthetical conclusion follows from the above remarks. Also by the above remarks, it suffices to show that $\Omega(M)/R$ is P -torsion for all prime ideals $P \neq M$. In other words, it

suffices to show that $(R : q) \not\subseteq P$ for all $q \in \Omega(M) \setminus R$. By the above description of $\Omega(M)$, we have $\text{Rad}(R : q) = M$, from which the assertion is clear. \square

The next two lemmas will be used in the proof of Proposition 3.6.

Lemma 3.4. *Let R be a ring such that Q , the total quotient ring of R , is von Neumann regular. Let I be an ideal of R and let x be a regular element of R/I . Then some coset representative of x is a regular element of R ; that is, $x = b + I$ for some regular element b of R .*

Proof. Let $a \in R$ be any coset representative of x . Since $\text{tq}(R)$ is von Neumann regular, there exists $s \in R$ such that $as = 0$ and $a + s$ is a regular element of R (cf. [1, Theorem 2.3]). Since $x \cdot (s + I) = 0 \in R/I$ and x is regular, we have $s + I = 0 \in R/I$. It follows that $(a + s) + I = a + I = x$, and so $b := a + s$ suffices. \square

Lemma 3.5. *Let R be a ring such that Q , the total quotient ring of R , is von Neumann regular. Let S be a multiplicatively closed subset of R . Then the total quotient ring of R_S is isomorphic to Q_S and, hence, is also von Neumann regular. In particular, if T is an overring of R , then T_S is (R -algebra isomorphic to) an overring of R_S .*

Proof. By taking $A := R$ and $B := R_S$, the assertion follows immediately from [18, Lemme 2.5]: if $\text{tq}(A)$ is a von Neumann regular ring and $A \rightarrow B$ is a unital ring homomorphism such that B is flat over both A and $B \otimes_A B$, then $\text{tq}(B)$ is a von Neumann regular ring and the canonical ring homomorphism $\text{tq}(A) \otimes_A B \rightarrow \text{tq}(B)$ is an isomorphism, since this map is an A -algebra homomorphism. The interested reader is invited to find an alternate proof based on [21, Propositions 2.4]. \square

Lemma 3.3 left open the nature of the localization of $\Omega(M)$ at M . This issue is addressed in Proposition 3.6, which is essentially a modification of [2, Proposition 2.3]. Following [2], we say that an ideal is of *finite type* if it is the radical of a finitely generated ideal.

Proposition 3.6. *Let R be a ring such that Q , the total quotient ring of R , is von Neumann regular. Let $M \in \text{Max}(R)$ such that R_M has a minimal overring of the form R_P for some prime ideal $P \subset M$. Let $\Omega(M)$ denote the Kaplansky transform of R with respect to M . Then the following conditions are equivalent:*

- (1) $\Omega(M) \neq R$;
- (2) $\Omega(M)_M = R_P$;
- (3) M is the unique element W of $\text{Spec}(R)$ such that $W\Omega(M) = \Omega(M)$;
- (4) M is of finite type.

Proof. Note that $M \notin \text{Min}(R)$ (after all, $P \subset M$). It therefore follows from the comments preceding Lemma 3.3 that $\Omega(M) = \{q \in Q \mid \text{Rad}(R : q) = M\} \cup R$ is an overring of R .

Observe that $R_P \cong (R_M)_{PR_M}$ is not integral over R_M . As $R_M \hookrightarrow R_P$ is a minimal ring extension, it must therefore be the case that R_M is integrally closed in R_P . Moreover, $MR_M \notin \text{Min}(R_M)$ since $PR_M \subset MR_M$. Hence, by Theorem 3.1, PR_M is a divided prime ideal of R_M and R_M/PR_M is a one-dimensional valuation domain. The one-dimensionality implies that there is no prime ideal of R contained strictly between P and M .

(1) \Rightarrow (2): Assume (1). By Lemma 3.3, $R_W = \Omega(M)_W$ for all $W \in \text{Spec}(R) \setminus \{M\}$. If $q \in \Omega(M) \setminus R$, then $\text{Rad}(R : q) = M$. Thus, $(R :_R q) \cap (R \setminus M) = \emptyset$, and so $\Omega(M)/R$ is M -torsion-free. Hence, the canonical map $\Omega(M)/R \rightarrow (\Omega(M)/R)_M \cong \Omega(M)_M/R_M$ is an injection. Since $\Omega(M) \neq R$, it follows that the inclusion map $R_M \hookrightarrow \Omega(M)_M$ is not surjective. Moreover, since $\Omega(M)_P = R_P$, we have that $\Omega(M)_M/R_M = (\Omega(M)/R)_M$ is P -torsion (since $(\Omega(M)_M)_P = (R_M)_P$ canonically).

As $\Omega(M)_M/R_M$ is P -torsion, we can identify $(R_M)_P = (\Omega(M)_M)_P$. Therefore, since $(R_M)_P \cong R_P$ canonically, there exists an R_M -algebra homomorphism $f : \Omega(M)_M \rightarrow R_P$. Also, since $\Omega(M)$ is an overring of R , Lemma 3.5 ensures that $\Omega(M)_M$ is an overring of R_M . Consequently, $\Omega(M)_M$ is an essential extension of R_M : this means that each nonzero R_M -submodule of $\Omega(M)_M$ has a nonzero intersection with R_M . Hence, f is an injection (for, otherwise, $\ker(f)$ would have a nonzero intersection with R_M , contradicting the fact that the canonical R_M -algebra map $R_M \rightarrow R_P$ is an injection). Identifying via f , we can thus consider the chain $R_M \subseteq \Omega(M)_M \subseteq R_P$. Since R_P is a minimal ring extension of R_M and we have seen that $R_M \subset \Omega(M)_M$, it follows that $\Omega(M)_M = R_P$.

(2) \Rightarrow (3): Assume (2). We show first that $M\Omega(M) = \Omega(M)$. Suppose not. Then the ideal $M\Omega(M)$ of $\Omega(M)$ is contained in some maximal ideal, say M' , of $\Omega(M)$. As $M' \cap R \supseteq M$, the maximality of M yields that $M' \cap R = M$. It follows that $M'\Omega(M)_M$ is a prime ideal of $\Omega(M)_M = R_P$, and so $M' \subseteq PR_P$. Taking inverse images in R , we find that

$$M = M' \cap R \subseteq PR_P \cap R = P \subset M,$$

the desired contradiction. Thus, $M\Omega(M) = \Omega(M)$.

It remains to show that if $W \in \text{Spec}(R)$ and $W \neq M$, then $W\Omega(M) \neq \Omega(M)$. Assume, on the contrary, that $W\Omega(M)_W = \Omega(M)_W$. However, $\Omega(M)_W = R_W$ by Lemma 3.3. Putting these two equations together tells us that $WR_W = R_W$, a contradiction.

(3) \Rightarrow (4): Since $M\Omega(M) = \Omega(M)$, there exist $m_1, m_2, \dots, m_r \in M$ and $t_1, \dots, t_r \in \Omega(M)$ such that

$$m_1 t_1 + m_2 t_2 + \dots + m_r t_r = 1.$$

Let I be the ideal of R that is generated by the finite set $\{m_1, m_2, \dots, m_r\}$. Then $I\Omega(M) = \Omega(M)$. Thus, $W\Omega(M) = \Omega(M)$ for each $W \in \text{Spec}(R)$ such that $I \subseteq W$. By (3), the only such W is M , and so $\text{Rad}(I) = M$. In accordance with the above definition, M is of finite type.

(4) \Rightarrow (1): By Theorem 3.1 and the second paragraph of this proof, $MR_M = \text{Rad}(xR_M)$ for some $x \in MR_M \setminus PR_M$. In particular, MR_M is the only prime ideal of R_M that contains x . Moreover, $MR_M \notin \text{Min}(R_M)$ since $PR_M \subset MR_M$.

As a subring of a von Neumann regular ring, R must be a reduced ring, and hence so is R_M . Thus, the set of zero-divisors in R_M is the union of the minimal prime ideals of R_M [16, Corollary 2.4]. It follows that x must be a regular element of R_M . Without loss of generality, $x = a/1$ for some $a \in M \setminus P \subset R$. In other words, we can view $x \in R/J$, where J denotes the kernel of the canonical ring homomorphism $g : R \rightarrow R_M$. Note that x is a regular element of R/J , since x is regular in (the larger ring) R_M . As $\text{tq}(R_M)$ is von Neumann regular (cf. Lemma 3.5), it now follows from Lemma 3.4 that $x = g(b)$ for some regular element b of R . Replacing a with b (but still calling it a by *abus de langage*), we can thus assume that $x = a/1$ where a is a regular element of R . (Notice that the change of notation still satisfies that $a \in M \setminus P \subset R$, since $x \in MR_M \setminus PR_M$.)

Assume (4), and thus choose a finitely generated ideal I of R such that $\text{Rad}(I) = M$. Therefore, $\text{Rad}(IR_M) = MR_M = \text{Rad}(xR_M)$. Since IR_M is a finitely generated ideal of R_M , it follows that there exists a positive integer k such that $I^k R_M \subseteq xR_M$. Furthermore, I^k is a finitely generated ideal of R , and so we can write $I^k = \sum_{i=1}^n Ra_i$ for some $a_1, a_2, \dots, a_n \in R$. Now, for each $i = 1, 2, \dots, n$, we have $a_i \in xR_M = aR_M$, and so there exist $s_i \in R \setminus M$ and $r_i \in R$ such that $a_i s_i = ar_i$. Put $s := s_1 \cdot s_2 \cdot \dots \cdot s_n$. Of course, $s \in R \setminus M$. Moreover, $s/a \notin R$ (for otherwise, $s = (s/a)a \in Ra \subseteq M$, a contradiction). It therefore suffices to prove that $s/a \in \Omega(M)$, for then the presence of s/a shows that $\Omega(M) \neq R$. To that end, it suffices to prove that $\text{Rad}(R : (s/a)) = M$, in view of the earlier description of $\Omega(M)$.

For each $i = 1, 2, \dots, n$, we have $a_i s = ar_i(s/s_i)$ and $s/s_i \in R$. Thus, $(s/a)a_i = (a_i s)/a \in R$ for each i . We conclude that $(s/a)I^k \subseteq R$, whence $I \subseteq \text{Rad}(R : (s/a))$. It follows that $M = \text{Rad}(I) \subseteq \text{Rad}(R : (s/a)) \subset R$, the last inclusion being proper since $s/a \notin R$. Hence, since M is maximal, $\text{Rad}(R : (s/a)) = M$, as desired. \square

We next present the main result of the section. Theorem 3.7 characterizes the existence of minimal ring extensions T in which a given base ring R having a von Neumann regular total quotient ring is integrally closed. This result generalizes [2, Theorem 2.4], which handled the case of R an integrally closed domain which is not a field and T (assumed to be) an overring of R .

Theorem 3.7. *Let R be a ring such that the total quotient ring, Q , of R is von Neumann regular. Then:*

- (a) *Let T be an extension ring of R and let M be a maximal ideal of R . Then the following three conditions are equivalent:*
 - (1) *T is a minimal ring extension of R such that R is integrally closed in T and M is the crucial maximal ideal of $R \subset T$;*
 - (2) *T is a minimal overring of R such that R is integrally closed in T and M is the crucial maximal ideal of $R \subset T$;*
 - (3) *The following three conditions hold:*
 - (i) *M is of finite type,*
 - (ii) *there exists a prime ideal $P \subset M$ such that PR_M is a divided prime ideal of R_M , there exists a regular element $u \in MR_M \setminus PR_M$, and $(R/P)_{M/P}$ is a one-dimensional valuation domain (hence, M is not a minimal prime ideal of R), and*
 - (iii) *$T \cong \Omega(M)$ as R -algebras.*
- (b) *The class consisting of the R -algebra isomorphism classes of minimal ring extensions of R in which R is integrally closed is in one-to-one correspondence with the set of maximal ideals M of R that satisfy conditions (i) and (ii) of condition (3) above.*

Proof. (a) (2) \Rightarrow (1): Trivial.

(1) \Rightarrow (3): Assume (1). Then, since T is not integral over R , it follows from [11, Théorème 2.2(iii)] that $R \hookrightarrow T$ is a flat epimorphism. Therefore, since $\text{tq}(R)$ is von Neumann regular, T is (R -algebra isomorphic to) an overring of R (cf. [15, Theorem 4.3.7]). It follows from Lemma 3.5 that T_M is (isomorphic to) an overring of R_M . Furthermore, as M is the crucial maximal ideal of $R \subset T$, [19, Proposition 3.5] and [5, Theorem 2.13.2] combine to yield that $P := (R : T)$ is a prime ideal of R such that R/P is a one-dimensional valuation domain with maximal ideal M/P and with quotient field T/P . In addition, $R_M \subset T_M$ inherits the property

of being a minimal ring extension from $R \subset T$. Thus, T_M is a minimal overring of R_M . It is also clear that $R_M \hookrightarrow T_M$ inherits the property of being a flat epimorphism from $R \hookrightarrow T$; and R_M is integrally closed in T_M (cf. [18, Corollaire 2.6]; or combine [19, Proposition 3.7] and [11, Théorème 2.2(iii)]). Thus, by the proof of Theorem 3.1, PR_M is a divided prime ideal of R_M and each element of $MR_M \setminus PR_M$ is a regular element of R_M . This completes the proof that condition (ii) holds.

It remains to establish (iii) and (i), namely, that $T = \Omega(M)$ and that M is of finite type. Of course, T is finitely generated as an R -algebra, since $R \subset T$ is a minimal ring extension. Since we also have that Q is von Neumann regular and T is a flat overring of R , it follows from [21, Corollary 3.9] that $T = \Omega(I)$ for some (finitely generated and regular) ideal I of R such that $IT = T$. Let $q \in T \setminus R$. By definition of the (generalized) Kaplansky transform, $I \subseteq \text{Rad}(R : q)$. Since $P = (R : T)$, we also have that $P \subseteq (R : q) \subseteq \text{Rad}(R : q)$, and so $I + P \subseteq \text{Rad}(R : q)$. Similarly, we see, for each $q \in Q$, that $I \subseteq \text{Rad}(R : q)$ if and only if $I + P \subseteq \text{Rad}(R : q)$. Using the definition of the Kaplansky transform, we can conclude that

$$T = \Omega(I) = \Omega(I + P) = \Omega(\text{Rad}(I + P)).$$

As P is a proper ideal of $T = \Omega(I)$ and $IT = T$, it follows that $I \not\subseteq P$. Hence, $P \subset I + P$. Note that $I + P \neq R$ since $\Omega(I + P) = T \neq R = \Omega(R)$. In fact, $I + P \subseteq M$ since $I \subseteq M$. (We indicate two ways to see this. First, choose $N \in \text{Max}(R)$ such that $I + P \subseteq N$. As $(R/P, M/P)$ is a one-dimensional quasilocal domain, M is the only prime ideal of R that contains P , and so $N = M$. For the second proof, suppose not, with $i \in I \setminus M$. Let $q \in T$. As $I \subseteq \text{Rad}(R : q)$, there exists a positive integer n such that $i^n q \in R$, whence $q \in R_i \subseteq R_M$. It follows that $T \subseteq R_M$ and $T_M = R_M$, contradicting the fact that the crucial maximal ideal M satisfies $R_M \subset T_M$.) Consequently $\text{Rad}(I + P) = M$, whence $T = \Omega(M)$, to complete the proof of (iii).

Having established (ii), we can apply Theorem 3.1 to obtain identifications $T_M = (R_M)_{PR_M} = R_P$. Thus, by (iii), $\Omega(M)_M = R_P$. Therefore, by Proposition 3.6 (which applies since $\text{tq}(R_M)$ is von Neumann regular), M is of finite type, thus proving (ii).

(3) \Rightarrow (2): Assume (3). As noted earlier, M contains a regular element (see the second paragraph of the proof of Theorem 2.4) and so, by an earlier comment, $\Omega(M)$ is an overring of R . Note, by Theorem 3.1, that $R_P = (R_M)_{PR_M}$ is a minimal overring of R_M and that R_M is integrally closed in R_P . Since M is, by hypothesis, of finite type, Proposition 3.6 gives the identifications $R_P = T_M = \Omega(M)_M$. In particular, $\Omega(M)_M \neq R_M$. Furthermore, by Lemma 3.3, we have that $R_N = \Omega(M)_N$ for each $N \in \text{Spec}(R)$ such that $N \neq M$. It follows that $\Omega(M)$ is a minimal ring extension of R (for any ring between R and $\Omega(M)$ is locally equal, hence equal, to either R or $\Omega(M)$); and that M is, by definition, the crucial maximal ideal of this minimal ring extension. It therefore suffices to show that R is integrally closed in $\Omega(M)$. If this were not the case, then $\Omega(M)$ would be integral over R , whence $\Omega(M)_M$ would be integral over R_M . However, we have seen that R_M is integrally closed in, and unequal to, T_M . The proof of (a) is complete.

(b) Let $[T]$ denote the R -algebra isomorphism class of an R -algebra T . Then the asserted one-to-one correspondence can be obtained by sending $[T]$ to the crucial maximal ideal of $R \subset T$; and by sending M (satisfying conditions (i) and (ii) above) to $[\Omega(M)]$. It follows from the equivalence (1) \Leftrightarrow (3) in (a) that these assignments are inverse to one another. \square

An amusing consequence of Theorem 3.7 is that if R is a von Neumann regular ring, then there does not exist a minimal ring extension T of R such that R is integrally closed in T . This

can be proved directly as follows. Since R is locally a field, we can combine the classification in Lemma 2.9 with [11, Lemme 1.3] to show that each minimal ring extension of R is (locally) integral over R .

Remark 3.8. (a) The known classification results permit an ideal-theoretic interpretation in some cases. For instance, let R be a Prüfer domain with only finitely many prime ideals. Then the R -algebra isomorphism classes of the minimal ring extensions of R form a set which is in one-to-one correspondence with the disjoint union of three copies of $\text{Max}(R)$. To see this, note first that Lemma 2.9 takes care of the assertion if R is a field. Assume next that R is not a field. Then the first of the asserted copies of $\text{Max}(R)$ arises (via Theorem 3.7(b)) from the minimal overrings of R , since R is integrally closed and each maximal ideal M of R satisfies conditions (i) and (ii) in the statement of Theorem 3.7 (cf. [2, Lemma 5.3]). The remaining two copies of $\text{Max}(R)$ arise (via [10, Theorem 2.7]) from the idealizations $R(+)R/M$ and the direct products $R \times R/M$.

(b) The construction of the one-to-one correspondence in Theorem 3.7(b) raises the following question. If M and N are regular maximal ideals of a ring R such that $\Omega(M) = \Omega(N)$, must it be the case that $M = N$? Theorem 3.7(b) gives an affirmative answer if $\text{tq}(R)$ is von Neumann regular and M, N satisfy conditions (i) and (ii) in the statement of Theorem 3.7. More generally, we easily get an affirmative answer from the definition of the generalized Kaplansky transform if $\Omega(M) = \Omega(N) \neq R$ (for if $q \in \text{tq}(R) \setminus R$, then $M = \text{Rad}(R : q) = N$). For a negative answer, it would suffice (in view of Proposition 3.6) to produce a domain R having (at least) two distinct maximal ideals neither of which is of finite type.

We next collect some observations pertaining to Theorem 3.1. Although Remark 3.9 could have immediately followed Theorem 3.1, it was deferred in order to hasten our path to Theorem 3.7.

Remark 3.9. (a) Theorem 3.1 is compatible with the result of Ferrand–Olivier that was stated in Lemma 2.9. Indeed, if R is a field, then being zero-dimensional, it cannot have a prime ideal P satisfying condition (2) in Theorem 3.1. Hence, by condition (1) in Theorem 3.1, each minimal ring extension of R is integral over R . This conclusion is corroborated by inspecting the extensions listed in the classification in Lemma 2.9.

(b) Reasoning via Theorem 3.1 as in (a), we see that if R is any quasilocal zero-dimensional ring, then each minimal ring extension T of R is integral over R . Of course, no such T is R -algebra isomorphic to an overring of R since each zero-dimensional ring coincides with its total quotient ring (cf. [17, Theorem 84]).

(c) Recall that a ring R is said to be a *chained ring* if the set of ideals of R is linearly ordered by inclusion. It is easy to see that any chained ring is quasilocal; any homomorphic image of a chained ring is a chained ring; and a domain is a chained ring if and only if it is a valuation domain (possibly a field). Since each chained ring is a divided ring (in the sense of [3], that is, each of its prime ideals is divided), it seems natural to seek an application of Theorem 3.1 to chained rings. One such application is the following. If (R, M) is a one-dimensional chained ring whose minimal prime ideal is divided, then either (i) R is integrally closed and $\text{tq}(R)$ is a minimal ring extension of R or (ii) each element of M is a zero-divisor of R (in which case, $R = \text{tq}(R)$ is not a reduced ring).

For a proof, notice that Theorem 3.1 applies since R is quasilocal. Most of condition (2) in Theorem 3.1 is trivially satisfied, for the above comments show that R has a unique nonmaximal prime ideal, say P_0 ; P_0 is a divided prime ideal of R ; and R/P_0 is a one-dimensional valuation

domain. Thus, R is a valuation domain if and only if $P_0 = 0$, that is, if and only if R is reduced; this takes care of the parenthetical comment in (ii). Moreover, if (ii) fails, then condition (2) in Theorem 3.1 is satisfied and then Theorem 3.1 yields that $\text{tq}(R) = R_{P_0}$ is a minimal ring extension of R in which R is integrally closed, and so R is integrally closed.

(d) The most obvious example of a Noetherian ring satisfying the hypotheses and the equivalent conditions in the result in (c) is an arbitrary DVR (discrete rank 1 valuation domain). We next present a non-Noetherian (and non-domain) example of a ring that satisfies the hypotheses and the equivalent conditions in the result in (c).

Consider the pullback $R := \mathbb{Z}_{2\mathbb{Z}} \times_{\mathbb{Q}} \mathbb{Q}[X]/(X^2)$. (In the following analysis, one could replace $\mathbb{Z}_{2\mathbb{Z}} \subset \mathbb{Q}$ with $V \subset K$ where V is any DVR having quotient field K ; R is then replaced with $V \times_K K[X]/(X^2)$.) By the order-theoretic impact of a fundamental gluing result [12, Theorem 1.4], R is a quasilocal one-dimensional ring, say with maximal ideal M . The only other prime ideal of R is $P_0 := X\mathbb{Q}[X]/(X^2)$. Note that $\mathbb{Q}[X]/(X^2)$, the ring of dual numbers over \mathbb{Q} , is isomorphic to the idealization $\mathbb{Q}(+)\mathbb{Q}$ and can be additively written as $\mathbb{Q} \oplus \mathbb{Q}x$, where $x := X + (X^2)$. Then R can be viewed additively as $\mathbb{Z}_{2\mathbb{Z}} \oplus \mathbb{Q}x$; furthermore, $P_0 = \mathbb{Q}x$ and $M = 2\mathbb{Z}_{2\mathbb{Z}} \oplus \mathbb{Q}x$. In particular, R is not reduced, since it contains x and $x^2 = 0$. Moreover, one can check that the principal ideals of R all take one of the forms $\mathbb{Z}_{2\mathbb{Z}}qx$, $\mathbb{Z}_{2\mathbb{Z}}a \oplus \mathbb{Q}x$ for suitable elements $q \in \mathbb{Q}$, $a \in \mathbb{Z}_{2\mathbb{Z}}$. Therefore, since $\mathbb{Z}_{2\mathbb{Z}}$ is a valuation domain, it follows that R is a chained ring. However, R is not a Noetherian ring, since P_0 is not a finitely generated ideal of R (essentially because \mathbb{Q} is not a finitely generated $\mathbb{Z}_{2\mathbb{Z}}$ -module).

We next indicate one way to verify that P_0 is a divided prime ideal of R . A calculation verifies that P_0 coincides with the set of zero-divisors of R . Thus, each zero-divisor of R is nilpotent, and so $\text{tq}(R) = R_{P_0}$ by [6, Proposition 2.3(b)]. Moreover, by [4, Proposition 2.5(c)], showing that P_0 is divided is equivalent to showing that $P_0 R_{P_0} = P_0$. Calculations show that $\text{tq}(R) = \mathbb{Q}(+)\mathbb{Q}$ (in other words, $\mathbb{Q}[X]/(X^2)$) and that $P_0 R_{P_0} = 0 \oplus \mathbb{Q}x$. Hence, R satisfies the hypotheses of the result in (c).

Note R does not satisfy condition (ii) in (c) since no element of $M \setminus P_0$ is a zero-divisor in R . Hence, by the result in (c), R must satisfy condition (i). Thus, R is integrally closed and $\text{tq}(R) = \mathbb{Q}(+)\mathbb{Q}$ is a minimal ring extension of R (in which R is integrally closed).

Remark 3.9(d) raises the question of what one can say about the minimal overrings of a Noetherian local ring. Recently, Picavet and Picavet-L'Hermitte have announced the following result [19, Proposition 5.7] along these lines: if R is an integrally closed local Noetherian ring which is not a domain, then R does not have a minimal overring. This result is generalized in Proposition 3.11 by using Theorem 3.1. Proposition 3.11 also provides another explanation why the non-domain chained ring R in Remark 3.9(d) is not a Noetherian ring. Lemma 3.10 records a useful fact that seems to have escaped earlier notice.

Lemma 3.10. *Let P be a finitely generated divided nonmaximal prime ideal of a quasilocal ring R . Then $P = 0$ (and so R is a domain but not a field).*

Proof. Let M denote the maximal ideal of R . Since P is nonmaximal, we can pick $z \in M \setminus P$. As P is divided, it follows that $P \subset Rz$, and hence that $P = Pz$. Thus, since P is finitely generated, the “determinant trick” yields an element $a \in Rz \subseteq M$ such that $(1 - a)P = 0$. However, $1 - a$ is a unit of R , and the assertions are now immediate. \square

Proposition 3.11.

- (a) Let (R, M) be a local Noetherian ring. Then there exists a minimal ring extension T of R such that R is integrally closed in T if and only if R is a DVR. Moreover, if these equivalent conditions hold, then T is R -algebra isomorphic to the quotient field of R .
- (b) Let (R, M) be a local Noetherian ring and let T be a minimal overring of R . Then either T is integral over R or R is a DVR with quotient field T .
- (c) Let (R, M) be an integrally closed local Noetherian ring. Then there does not exist a minimal ring extension T of R such that $R \subset T \subset \text{tq}(R)$.

Proof. (a) If R is a DVR (hence, a one-dimensional valuation domain) with quotient field K , then K is a minimal ring extension of R in which R is integrally closed. Conversely, suppose that there exists a minimal ring extension T of R such that R is integrally closed in T . Then Theorem 3.1 supplies a divided prime ideal P of R such that R/P is a one-dimensional valuation domain and $T \cong R_P$. Note that P is finitely generated (since R is Noetherian) and nonmaximal (since R/P is not a field). Hence, by Lemma 3.10, $P = 0$ and R is a domain but not a field. Then $R \cong R/0$ is a Noetherian valuation domain which is not a field, that is, a DVR; and $T \cong R_0$, the quotient field of R .

(b) Since $R \subset T$ is a minimal ring extension, either T is integral over R or R is integrally closed in T . Hence, the assertion follows from (a).

(c) Suppose, on the contrary, that such T exists. Then T is not integral over R , and so by (b), $T = \text{tq}(R)$, contradicting the requirements on T . \square

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